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Ensemble inequivalence in systems with long-range interactions

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Abstract

Ensemble inequivalence has been observed in several systems. In particular it has been recently shown that negative specific heat can arise in the microcanonical ensemble in the thermodynamic limit for systems with long-range interactions.We display a connection between such behaviour and a mean-field like structure of the partition function. Since short-range models cannot display this kind of behaviour, this strongly suggests that such systems are necessarily non-mean field in the sense indicated here. We illustrate our results showing an application to the Blume–Emery–Griffiths model. We further show that a broad class of systems with non-integrable interactions are indeed of mean-field type in the sense specified, so that they are expected to display ensemble inequivalence as well as the peculiar behaviour described above in the microcanonical ensemble.

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1. Introduction

Particle or spin systems for which the pairwise interaction potential decays at large distances with a power smaller than the space dimension are called *long-range* or *non-integrable*. It has been suggested that at first-order phase transitions such systems should display *ensemble inequivalence* also in the thermodynamic limit [1, 2]. A few examples where this is explicitly shown, both analytically and numerically, have been published [1, 3–6]. The specific heat, which is always positive in the canonical ensemble, may become negative in the microcanonical ensemble, and even temperature jumps may appear when continuously varying the energy. Negative specific heat was first observed in gravitational systems during the process of

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gravothermal collapse, but here the situation is made more complex by the singularity of the interaction at short distances (for a review, see [7]). In this paper we present a formal approach within which we can easily prove ensemble inequivalence, and all the unexpected features of the microcanonical ensemble naturally arise. The approach is based on the assumption, which we justify afterwards, that a sort of Landau free energy can be always introduced for long-range systems and that it is endowed with good analyticity properties. A preliminary version of these results has been presented briefly in a workshop proceedings [8].

2. Mean-field and ensemble inequivalence

In this section, we show the main result of this paper. We say that a system is of *mean-field type* if it satisfies the following condition:

$$Z_C(\beta) = \int_{-\infty}^{\infty} \exp\left[-N\Psi(\beta, m)\right] \mathrm{d}m \tag{1}$$

where $Z_C(\beta)$ is the canonical partition function and $\Psi(\beta, m)$ is a real analytic function of β and m. Note that we require this for finite values of N, but that we additionally require the function $\Psi(\beta, m)$ to be independent of N. A weaker requirement might be that Ψ is, in some sense, well-behaved in the $N \to \infty$ limit as far as analyticity is concerned. However, we shall not pursue this line any further here. Note in particular that infinite range models always have this property, as follows from their solution using, for example, the Hubbard–Stratonovich transformation.

From this, it follows almost immediately that ensemble inequivalence will in general occur. Indeed, from (1) one obtains the following expression for the phase space volume of the energy shell, which may be viewed as a kind of microcanonical partition function

$$Z_M(\epsilon) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\lambda}{2\pi}\, \mathrm{e}^{\mathrm{i}\lambda N\epsilon} Z_C(\mathrm{i}\lambda) \tag{2}$$

where ϵ is the energy per particle. Using analyticity to rotate the integration contour, this can be recast in the following form, after inverting the order of integration

$$Z_M(\epsilon) = \int_{-\infty}^{\infty} \mathrm{d}m \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathrm{d}\lambda}{2\pi\mathrm{i}} \exp\left[N(\lambda\epsilon - \Psi(\lambda, m)\right]. \tag{3}$$

One can now perform a saddle point integral to estimate the value of the λ integral for large values of N. We argue that the dominant saddle point must lie on the real axis. If it were otherwise, we would, because of the reality properties of Ψ, have two complex conjugate dominant saddle points, and hence an oscillatory behaviour of the partition function. However, if the dominant behaviour of the partition function is oscillatory, it will generally take on at least some negative values, which is absurd. We may therefore limit ourselves to the consideration of real saddle points. Since the integration path is perpendicular to the real axis, however, the minima along the real λ axis of the argument of the exponential will correspond to maxima when traversed along the λ imaginary axis. We are therefore led to the following expression for the microcanonical partition function:

$$Z_M(\epsilon) = \int_{-\infty}^{\infty} \exp[N \min_{\lambda}(\lambda\epsilon - \Psi(\lambda, m))] = \exp[N \max_{m} \min_{\lambda}(\lambda\epsilon - \Psi(\lambda, m))]. \tag{4}$$

From this one immediately obtains for the entropy per particle

$$S_M(\epsilon) = \max_{m} \min_{\lambda}[\lambda\epsilon - \Psi(\lambda, m))]. \tag{5}$$

This result can now be compared with the standard result for the canonical ensemble, which is obtained as follows: the free energy per particle is found, after a straightforward estimate of (1) via Laplace's method, to be given by

$$F(\beta) = \beta^{-1} \min_{m} \Psi(\beta, m). \tag{6}$$

It is now readily seen, through standard thermodynamic identities, that the entropy is the Legendre transform of $\beta F(\beta)$ with respect to β. This then leads to the following expression for $S_{\mathrm{C}}(\epsilon)$:

$$S_C(\epsilon) = \min_{\lambda} \max_{m} [\lambda\epsilon - \Psi(\lambda, m))]. \tag{7}$$

From these results, a few consequences are immediate:

1. The two forms of the entropy need not be equal. Indeed, if we could prove that $\Psi(\lambda, m)$ has a unique extremum under certain conditions, we could argue for equality. However, we know that this is not generally the case: whenever a phase transition occurs, the function $\Psi(\lambda, m)$ may have multiple extrema at least as a function of m, thus precluding any simple statements about the identity of the two entropies.
2. Quite generally, however, one may say that

$$S_M(\epsilon) \leqslant S_C(\epsilon). \tag{8}$$

   This is a general property of such min–max combinations and is trivial to prove (see appendix A). It is, however, satisfying, since it suggests that in some sense the microcanonical entropy is a restricted entropy, and that the real reason why the two are different is that the microcanonical system cannot relax in certain ways, reaching the maximal possible entropy in the canonical ensemble.
3. Let us now consider the behaviour of the function $\Psi(\lambda, m)$ at the equilibrium points $\left(\lambda_c^\star, m_c^\star\right)$ and $\left(\lambda_m^\star, m_m^\star\right)$ for the canonical and microcanonical problems, respectively. For the canonical problem we find the following conditions for the point $\left(\lambda_c^\star, m_c^\star\right)$ to be the appropriate extremum

$$\frac{\partial^2\Psi}{\partial\lambda^2} < 0 \qquad \frac{\partial^2\Psi}{\partial m^2} > 0. \tag{9}$$

   These correspond quite naturally to the usual stability conditions for the thermodynamic potential $\Psi(\lambda, m)$, which is concave with respect to λ and convex with respect to m. However, if one goes through the same computation for the microcanonical ensemble, one finds

$$\frac{\partial^2\Psi}{\partial\lambda^2} < 0 \qquad \frac{\partial^2\Psi}{\partial m^2} - \frac{(\partial^2\Psi/\partial m\partial\lambda)^2}{\partial^2\Psi/\partial\lambda^2} > 0. \tag{10}$$

   Note in particular how this implies that the second derivative of Ψ with respect to m can take either sign. In fact, one verifies that the above two conditions have the following geometric meaning: The first indicates that Ψ is a local maximum in the λ direction, and the second amounts to stating that the matrix of second derivatives of Ψ has negative determinant, that is, the extremum must be of saddle-point type. These conditions are clearly weaker than (9), so that more extrema are allowed for the microcanonical ensemble than for the canonical.
4. If we consider the failure of concavity of the entropy as a function of ϵ, related to the findings concerning negative specific heats in certain systems, we again find that this can happen in the microcanonical case. Indeed, one sees immediately that this cannot occur in the canonical ensemble, since there the entropy is given as a minimum of linear functions,

which is of necessity concave. On the other hand, the microcanonical expression yields a maximum over concave functions, which need not be concave. More specifically, an explicit evaluation of the second derivative of $S_M(\epsilon)$ yields

$$\frac{\mathrm{d}^2 S_M}{\mathrm{d}\epsilon^2} = \frac{\partial^2\Psi}{\partial\lambda^2} - \frac{(\partial^2\Psi/\partial m\partial\lambda)^2}{\partial^2\Psi/\partial m^2}. \tag{11}$$

Again, since $\partial^2\Psi/\partial m^2$ can take both signs, one sees that the specific heat can do so as well. In fact, combining this result with (10), one finds that the sign of $\mathrm{d}^2 S_M/\mathrm{d}\epsilon^2$ is the opposite of that of $\partial^2\Psi/\partial m^2$. This means that the specific heat is negative exactly when the value of m corresponding to microcanonical equilibrium is unstable from the point of view of the canonical ensemble.

5. It has also been found that the temperature T can have a discontinuous dependence on ϵ in the microcanonical ensemble. Again this readily follows from our formalism: In the canonical ensemble, one effects a Legendre transformation of the following function

$$F(\lambda) = \max_m \Psi(\lambda, m). \tag{12}$$

Now F can have discontinuities in its derivative, but it follows from the analyticity of Ψ that it cannot have any straight line segments. From this follows through well-known considerations that the entropy can have straight line segments (corresponding to ordinary first-order phase transitions), but no jumps in its derivative. On the other hand, the microcanonical entropy $S_M(\epsilon)$ arises as the maximum over a set of concave functions, none of which can have jumps in the derivative. This clearly cannot exclude such jumps in $S_M(\epsilon)$, which have indeed been found in specific models.

6. Finally, we may note that since the equivalence between ensembles is rigorously proved for interactions which are of sufficiently short-range [9], this argument strongly suggests that such systems are by necessity non mean-field, that is, that such functions as Ψ do not exist for such systems, or, more precisely, do not have the required analyticity properties. The argument is not fully rigorous: As pointed out above, it is in principle possible to have such an analytic function and yet not to have any cases in which the two ensembles differ. This would in particular be the case if, for all physical parameter values, the function Ψ had only one extremum. However, it is certainly very suggestive.

3. Application to the Blume–Emery–Griffiths model

Let us show how the above approach works in the case of the Blume–Emery–Griffiths (BEG) model [10], which is a spin Hamiltonian on a lattice defined as follows:

$$H = \Delta \sum_{i=1}^{N} s_i^2 - \frac{J}{2N}\left(\sum_{i=1}^{N} s_i\right)^2 \tag{13}$$

where the s_i are spin variables taking the values 0, ± 1. Here, Δ and J are positive coupling constants. This model has a line of second-order phase transitions, which terminates in a tricritical point and is followed by a line of first-order phase transitions. This model, since it is in the mean field limit, can be solved exactly using the Hubbard–Stratonovich transformation, to yield a representation of the partition function of the form (1). One finds

$$\Psi(\beta, m) = \frac{J\beta m^2}{2} - \ln(1 + 2\mathrm{e}^{-\beta\Delta}\cosh(\beta J m)). \tag{14}$$

We now show that finding the extremum of $\lambda\epsilon - \Psi(\lambda, m)$ with respect to λ and m is equivalent to looking for the extrema of the entropy at given energy ϵ in the microcanonical ensemble. Indeed, if one defines

$$m = \frac{1}{N}\sum_{i=1}^{N} s_i \qquad q = \frac{1}{N}\sum_{i=1}^{N} s_i^2 \tag{15}$$

one readily finds that the entropy and the energy of (13) are given by

$$\begin{aligned} S_M(\epsilon) &= (1-q)\ln(1-q) + \frac{q+m}{2}\ln\frac{q+m}{2} + \frac{q-m}{2}\ln\frac{q-m}{2} \\ \epsilon &= \Delta q - \frac{Jm^2}{2}. \end{aligned} \tag{16}$$

If one now maximizes

$$S_M(\epsilon) + \mu\left(\epsilon - \Delta q + \frac{Jm^2}{2}\right) \tag{17}$$

one finds that this leads to the same values of μ and m as the maximization of $\lambda\epsilon - \Psi(\lambda, m)$ gives for λ and m, respectively. Further, the resulting values for the entropy are shown to be equal. These somewhat messy computations are performed in appendix B. It has been recently shown that in this model all the peculiarities mentioned in section 2 actually occur [5].

4. General behaviour at second-order phase transitions and tricritical points

In the preceding section, we have shown how, in the case of a specific model, our formalism was able to re-derive the usual microcanonical expressions. However, as these expressions are quite cumbersome, it is not possible to derive in a transparent and explicit way the exact nature of the various thermodynamic anomalies encountered there. In this section, we shall pursue a different approach. We shall consider quite general systems, but limit ourselves to the region in the vicinity of a phase transition. Under these circumstances, one can perform a Taylor expansion of the function Ψ as a low-order polynomial. This then allows a complete study of the stability conditions to be carried out.

To apply our formalism as developed in section 1, we require an approximate expression for the function

$$G(\lambda, m; \epsilon, \Delta) = \lambda\epsilon - \Psi(\lambda, m; \Delta) \tag{18}$$

where Δ represents one or more additional parameters of the Hamiltonian. Since we are interested in the vicinity of phase transitions, we are in fact concentrating on those regions in which new stationary points of G appear or disappear. Such regions are found in the vicinity of the solutions of the following equation (condition)

$$\det D^2G(\lambda, m; \epsilon, \Delta) = 0 \tag{19}$$

where D^2G means the matrix of second derivatives of G taken with respect to λ and m, and the variables ϵ and Δ are chosen so that λ and m are the corresponding stationary values.

Let us first look at the simplest case in which there are no additional parameters. If we further assume that the function G is symmetric in m with respect to change of sign, one obtains the following result: it is generically possible for two new minima to arise whenever the second derivative of G with respect to m vanishes at the origin. If we consider G near this point and measure all quantities with respect to the critical values, which we therefore set equal to zero, we obtain the following expression for G in the vicinity of the transition:

$$G(\lambda, m; \epsilon) = \frac{c_0\lambda^2}{2} + a_2\lambda m^2 - m^4 + \lambda\epsilon \tag{20}$$

where we have introduced an arbitrary numerical factor to make the coefficient of the m^4 term equal to unity and the minus sign is required by the thermodynamic stability of the system. The other constants are typically of order 1. New phenomena may indeed appear when these are of the same order as m, in which case, however, new parameters Δ should be introduced and the system should be embedded in one that has an appropriate multicritical point.

One now needs to test for the microcanonical stability conditions (10). The first one, which is also necessary for canonical stability, is equivalent to the positivity of c_0. The second one is written in terms of G as

$$\det D^2G = \det\begin{pmatrix} c_0 & 2a_2m \\ 2a_2m & 2a_2\lambda - 12m^2 \end{pmatrix} < 0 \tag{21}$$

and the equations expressing the stationarity of G with respect to variations of λ and m are given by

$$2a_2\lambda m - 4m^3 = 0 \tag{22}$$

$$c_0\lambda + a_2m^2 + \epsilon = 0. \tag{23}$$

Equation (22) determines λ once m is known, unless m is zero (paramagnetic phase), in which case (22) is a tautology. In this latter case, the condition (21) is equivalent to $a_2\lambda < 0$, since c_0 is always positive. However, this is just the condition that ensures the absence of non-zero real solutions of (22), so that we may say in summary that the paramagnetic phase is microcanonically stable only if there is no ferromagnetic solution.

On the other hand, if $m \neq 0$, it follows that

$$a_2\lambda = 2m^2 \tag{24}$$

from which it is immediate that (21) always holds good. Therefore, the ferromagnetic solution is always stable whenever it exists. For this reason, there is no possibility of ensemble inequivalence near a generic second-order phase transition.

The issue of tricritical points is a bit more subtle, since in this case we may have indeed ensemble inequivalence. The simplest case, to which we limit ourselves, is that in which a single additional parameter Δ is introduced and one considers the case where the second and the fourth derivatives of G vanish simultaneously when $m = 0$. We assume that this occurs at $\epsilon = \Delta = 0$, and we choose the value of λ which corresponds to the tricritical point to be zero as well. Again we assume G symmetric in m around zero. In this case the Taylor expansion is given by

$$G(\lambda, m; \epsilon, \Delta) = \frac{c_0\lambda^2}{2} + (a_2\lambda + b_2\Delta)m^2 + (a_4\lambda + b_4\Delta)m^4 - m^6 + \lambda\epsilon. \tag{25}$$

The equations determining λ and m are now

$$(a_2\lambda + b_2\Delta) + 2(a_4\lambda + b_4\Delta)m^2 - 3m^4 = 0 \tag{26}$$

$$c_0\lambda + a_2m^2 + a_4m^4 + \epsilon = 0. \tag{27}$$

Here (26) holds only if $m \neq 0$, otherwise it should be discarded. From (27) one obtains

$$\lambda = -\frac{a_2m^2 + \epsilon}{c_0} + O(m^4). \tag{28}$$

If one substitutes (28) into (25), one obtains a term of order m^4 with prefactor $-a_2^2/c_0$, which is of order 1 and is not compensated by any other term of similar order of magnitude. Thus we see that in the microcanonical ensemble the point corresponding to the canonical tricritical point is not tricritical anymore. Rather, since the m^4 term has negative sign, we do not need the m^6 term for stabilization anymore and we are led back to the case of second-order transitions discussed above. From this fact it immediately follows that ensemble equivalence cannot hold near canonical tricritical points. On the other hand, there may be *microcanonical* tricritical points, in which various microcanonical equilibria merge in the same way as in a canonical tricritical point. The study of such points is reserved to a later study.

5. Models with long-range interactions

In this section, we show that a large class of models having non-integrable interactions, actually satisfy our criterion (1) and can hence be expected to display the whole gamut of phenomena discussed in section 2. For the sake of definiteness we restrict ourselves to spin models on a lattice. Extensions to more general cases are presumably unproblematic. Consider the following Hamiltonian

$$H[s(\vec{i})] = L^{-(d-\alpha)} \sum_{\vec{k},\vec{l}} \frac{s(\vec{k})s(\vec{l})}{|\vec{k}-\vec{l}|^{\alpha}} + \sum_{\vec{k}} V[s(\vec{k})]. \tag{29}$$

Here the s are spins which run over a discrete set S, the indices $\vec{k}$ run over a d-dimensional lattice and α is an exponent between zero and d. The normalization of the interaction by $L^{-(d-\alpha)}$ guarantees that the Hamiltonian is in fact *extensive*. Note that the BEG model treated above, see (13), corresponds to a special case of (29), in which $\alpha = 0$.

To evaluate the partition function, we divide the volume into a large number of cells, such that the following conditions are satisfied: first, the cells are large enough that a partition function involving only the one-body terms can be evaluated accurately using saddle-point techniques; second, the cell should be small enough with respect to the whole sample, so that the interaction between the spins of one cell is negligible compared to the interaction with all other cells. Due to the non-integrable nature of the interaction, this can always be achieved by making the sample sufficiently large. We denote the centres of the cells by $\vec{x}$, $\vec{y}$ and introduce the coarse-grained variables

$$\rho(\vec{x}) = \frac{1}{v} \sum_{\vec{k}\in C(\vec{x})} s(\vec{k}). \tag{30}$$

Here $C(\vec{x})$ denotes the cell at $\vec{x}$ and v its volume. We further define $\Psi(\beta, \rho)$ for a system of volume v by

$$\exp[-N\Psi(\beta,\rho)] = \sum_{s(\vec{k})} \delta\left[\frac{1}{v}\sum_{\vec{k}} s(\vec{k}) - \rho\right] \exp\left[-\beta \sum_{\vec{k}} V\left(s(\vec{k})\right)\right]. \tag{31}$$

In order to express the partition function in terms of the coarse-grained variables, we need an expression for the interaction energy in terms of $\rho(\vec{x})$. This is obtained by the following consideration: as a consequence of the normalization, the interaction between any spin and those of its own cell or of nearby cells can be neglected. Since one only needs to consider distant spins, it is sufficient to use the first term of a Taylor expansion for the energy, which leads to the expression

$$E = L^{-(d-\alpha)} \sum_{\vec{x},\vec{y}} \frac{\rho(\vec{x})\rho(\vec{y})}{|\vec{x}-\vec{y}|^{\alpha}}. \tag{32}$$

From this one obtains the following expression for the partition function:

$$\sum_{\rho(\vec{x})} \exp\left[-\frac{\beta}{L^{d-\alpha}} \sum_{\vec{x},\vec{y}} \frac{\rho(\vec{x})\rho(\vec{y})}{|\vec{x}-\vec{y}]^{\alpha}} - N\sum_{\vec{x}} \Psi(\beta, \rho(\vec{x}))\right]. \tag{33}$$

Here the sum extends over all functions $\rho(\vec{x})$ which have integral N, where N is the total number of cells. This can now be evaluated straightforwardly using the Laplace method. The main contribution therefore comes from solutions of the equation

$$\frac{\partial \Psi}{\partial \rho}(\beta, \rho(\vec{x})) = -\frac{\beta}{L^{d-\alpha}} \sum_{\vec{y}} \frac{\rho(\vec{y})}{|\vec{x} - \vec{y}|^{\alpha}}. \tag{34}$$

We now show that the right-hand side is independent of $\vec{x}$. From this it follows that the dominant contributions come from constant density profiles. One can therefore replace the functional integration in (33) by an ordinary integration over the constant value of $\rho(\vec{x})$ and one has cast the partition function in the form (1).

To show the constancy of the right-hand side of (34), it is enough to note the following: the difference between the right-hand side evaluated at $\vec{x}$ and evaluated at $\vec{x} + \vec{a}$ contains two types of terms. First, contributions due to cells near $\vec{x}$ or $\vec{x} + \vec{a}$. These give a finite number of contributions of order 1 to the sum, and no contribution to the right-hand side itself, because of the normalization. Secondly, one has distant terms, the contribution of which is estimated by a Taylor expansion. However, it is clear that in leading order, these terms are identical for $\vec{x}$ and $\vec{x} + \vec{a}$, thus showing the result.

6. Conclusions and perspectives

Having shown that the mean-field formalism analysed in section 2 already contains all the peculiarities of ensemble inequivalence, a first future direction of research might be to obtain exact, approximate or even numerical solutions of well-chosen models with slowly decaying interactions, of the type discussed in section 5. The aim will be to test the behaviour of physical quantities, such as the specific heat, as the exponent of the decaying interaction is varied from long-range to short-range coupling, similar to what has been done for coupled rotators models [11, 12]. A related issue is the study of the behaviour of finite systems as a function of the ratio between the system size and the range of interaction. A further direction of research has to do with non-equilibrium properties. For instance, one might ask what happens when two systems of comparable size, both having negative specific heat, are put in contact with each other. This removes the energy conservation constraint of the two systems. What is the behaviour of the full system in this case? We might expect the microcanonical entropy to come closer to the canonical, although remaining bounded from above by it. This would yield the paradoxical result that the coupling of two systems with identical intensive parameters leads to an irreversible increase of entropy.

Finally, we have not mentioned at all in this paper non-equilibrium dynamical properties of long-range systems, which might well be analysed in the context of the thermodynamic formalism introduced here. For instance, the coherent clustering behaviour observed in the antiferromagnetic mean-field *XY* model [13], and recently found to be a non-equilibrium phenomenon [14], might well find some explanation in the context of the current formalism.

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Appendix A

In this appendix, we show quite generally the inequality

$$\max_x \min_y f(x, y) \leqslant \min_y \max_x f(x, y) \tag{A1}$$

from which (8) follows immediately. Assume that the two extrema are attained at the two points (x_1, y_1) and (x_2, y_2), respectively, that is

$$f(x_1, y_1) \leqslant f(x_1, y) \qquad f(x_2, y_2) \geqslant f(x, y_2). \tag{A2}$$

From this follows

$$f(x_1, y_1) \leqslant f(x_1, y_2) \leqslant f(x_2, y_2) \tag{A3}$$

which implies the inequality (A1).

Appendix B

Here we complete the proof of the equivalence between the two ways of computing the microcanonical entropy of the BEG model, by performing the missing computations. The equations arising from the minimization of (14) are

$$\begin{aligned} &\epsilon - \frac{Jm^2}{2} + 2\frac{-\Delta e^{-\Delta\lambda}\cosh Jm\lambda + Jm e^{-\Delta\lambda}\sinh Jm\lambda}{1 + 2e^{-\Delta\lambda}\cosh Jm\lambda} = 0 \\ &-m + 2\frac{e^{-\Delta\lambda}\sinh Jm\lambda}{1 + 2e^{-\Delta\lambda}\cosh Jm\lambda} = 0 \end{aligned} \tag{B1}$$

which are readily combined to yield

$$\begin{aligned} &m = 2\frac{e^{-\Delta\lambda}\sinh Jm\lambda}{1 + 2e^{-\Delta\lambda}\cosh Jm\lambda} \\ &\epsilon + \frac{Jm^2}{2} - \Delta e^{-\Delta\lambda} m \coth Jm\lambda = 0. \end{aligned} \tag{B2}$$

On the other hand, finding the extrema of (17) leads to the following conditions:

$$\ln(1-q) + \ln\frac{q+m}{2} + \ln\frac{q-m}{2} - \mu\Delta = 0 \tag{B3}$$

$$\ln\frac{q+m}{2} - \ln\frac{q-m}{2} + 2J\mu m = 0 \tag{B4}$$

$$\epsilon + \frac{Jm^2}{2} - \Delta q = 0. \tag{B5}$$

From (B4) one obtains

$$q = m \coth Jm\mu. \tag{B6}$$

If one now substitutes (B6) into (B3) and (B5), one obtains the equations (B2), where λ must be identified with μ. We therefore see that the two sets of equations are fully equivalent. The verification that the value of the entropy is the same in both cases is now straightforward algebra.

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